Unicity in Approximation of a Function and its Derivatives

By Lee Johnson

For f continuous and real on $[0, 1]$, let $||f|| = \max |f(x)|$, $x \in [0, 1]$. In this journal, Moursund [3] proved

THEOREM 1. *Let f be twice differentiable on* $[0, 1]$. *Among all polynomials $h(x)$ of degree n or less, let $p(x)$ be the one that minimizes*: $\max \{||h - f||, ||h' - f'||\}$. *If $q(x)$ is another such minimizing polynomial, then $q' = p'$.*

Let f^i denote the ith derivative of f. Moursund's result can be extended to:

THEOREM 2. *Let f be $(k + 1)$-times differentiable on* $[0, 1]$. *Among all polynomials $h(x)$ of degree n or less, let $p(x)$ be the one that minimizes*:

$$\max \{||h - f||, ||h^1 - f^1||, \cdots, ||h^k - f^k||\}.$$

If $q(x)$ is another such minimizing polynomial, then $q^k = p^k$.

We need some preliminary results before establishing Theorem 2. Let $M(h) = \max \{||h||, \cdots, ||h^k||\}$. The functional M is a norm on the set S of functions that are $(k + 1)$-times differentiable on $[0, 1]$.

Let Q denote the set of polynomials of degree n or less. Call $p_0 \in Q$ a best approximation to $f \in S$ if $M(p_0 - f) \leqq M(q - f)$, for all $q \in Q$. It can be shown [1] that the set of best approximations is convex and nonempty.

Call $x \in [0, 1]$ an extreme point of $p - f$ if for some i, $0 \leqq i \leqq k$, $|p^i(x) - f^i(x)| = ||p^i - f^i|| = M(p - f)$. Denote the set of extreme points of $p - f$ by $E(p, f)$. Standard arguments quickly show [2] that p is a best approximation to f if and only if p is a best approximation to f on $E(p, f)$.

Proof of Theorem 2. Let p and q be two best approximations to f; and suppose $p^k \neq q^k$. Let $c = tp + (1 - t)q$, $t \in (0, 1)$; then c is also a best approximation to f. Using $p^k \neq q^k$, we will construct an approximation to f that is better than c on $E(c, f)$, giving a contradiction. Let $a_i = j$ if there are j points x in $(0, 1)$ such that $|c^i(x) - f^i(x)| = ||c^i - f^i|| = M(c - f)$.

Let $b_i = 0, 1, 2$ according as none, one or both of $z = 0$, $z = 1$ are such that $|c^i(z) - f^i(z)| = M(c - f)$. In particular, $a_i = b_i = 0$ if $||c^i - f^i|| < M(c - f)$.

If x_0 is among the a_i extreme points of $c^i - f^i$, then

(1) x_0 is not among the a_{i+1} extreme points of $c^{i+1} - f^{i+1}$,

(2) $p^i(x_0) - f^i(x_0) = q^i(x_0) - f^i(x_0) = \pm M(c - f)$,

(3) $p^{i+1}(x_0) - f^{i+1}(x_0) = q^{i+1}(x_0) - f^{i+1}(x_0) = 0$.

From (2) and (3), $p^i(x) - q^i(x)$ has at least $2a_i + b_i$ zeroes. We will show that $p^i - q^i$ has at least $(b_0 + \cdots + b_i) + 2(a_0 + \cdots + a_i) - i$ zeroes.

LEMMA. *Let $h(x)$ be a polynomial with r single zeroes, s double zeroes and t triple zeroes. Let $h'(x)$ have u double zeroes—none of which are among the t triple zeroes of $h(x)$. Then $h'(x)$ has at least $r + 2s + 3t + 2u - 1$ zeroes.*

Proof. Let $r + s + t = v$, and label the zeroes of $h(x)$ as $x_1, \cdots, x_v$; $x_i < x_{i+1}$.

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In (x_i, x_{i+1}) there is a zero of $h'(x)$; furthermore, this zero must be of odd multiplicity. Also none of the u double zeroes of $h'(x)$ are counted among the v distinct zeroes of $h(x)$. Counting the zeroes of $h'(x)$ we obtain

(a) $s + 2t$; from the multiple zeroes of $h(x)$,

(b) $v - 1$; the zeroes of $h'(x)$ in (x_i, x_{i+1}),

(c) $2u$; as noted, the $v - 1$ zeroes in (b) are of odd multiplicity. If one of the u double zeroes of $h'(x)$ is included in (b), this zero must have been of multiplicity 3 or more.

Adding (a), (b) and (c), establishes the lemma.

Using (1), (2) and (3) from above; and applying the lemma repeatedly to the derivatives of $p(x) - q(x)$, we obtain

(4) $p^i(x) - q^i(x)$ has at least $(b_0 + \cdots + b_i) + 2(a_0 + \cdots + a_i) - i$ zeroes.

As $p^k - q^k \neq 0$, it must be that $n - k \geqq (b_0 + \cdots + b_k) + 2(a_0 + \cdots + a_k) - k$.

The same argument, starting with $p^j(x) - q^j(x)$, gives that $p^k - q^k$ has $(b_j + \cdots + b_k) + 2(a_j + \cdots + a_k) - (k - j)$ zeroes. Thus, $p^k \neq q^k$ means that

(5) $(b_j + \cdots + b_k) + 2(a_j + \cdots + a_k) \leqq n - j,\ 0 \leqq j \leqq k.$

We will use (5) to construct a polynomial $r(x)$ such that $r^i(y) - f^i(y) = 0$ if y is one of the $a_i + b_i$ extreme points of $c^i - f^i$. Select s points in (0, 1), distinct from the $a_0 + b_0$ extreme points of $c - f$; where $s + (b_0 + \cdots + b_k) + 2(a_0 + \cdots + a_k) = n + 1$. Note that from (5), $s \geqq 1$.

Let $D(x) = (1, x, x^2, \cdots, x^n)$, $D^1(x) = (0, 1, 2x, \cdots, nx^{n-1})$, $D^2(x) = (0, 0, 2, \cdots, n(n-1)x^{n-2})$. Define D^i similarly, $i = 3, \cdots, k$. We will form an $(n + 1) \times (n + 1)$ "Vandermonde-like" matrix A, as follows. For each of the s points $y_1, \cdots, y_s$ chosen in (0, 1), let A have a row of the form $D(y_i)$. For each of the a_0 extreme points w of $c - f$, let A have two rows of the form

$$\begin{matrix} D(w) \\ D'(w)\,. \end{matrix}$$

For each of the b_0 "end-point" extreme points z of $c - f$, let A have a row of the form $D(z)$.

Generally, for each of the a_i extreme points w of $c^i - f^i$, let A have 2 rows of the form

$$\begin{matrix} D^i(w) \\ D^{i+1}(w)\,. \end{matrix}$$

For each of the b_i "end-point" extreme points of $c^i - f^i$, let A have a row of the form $D^i(z)$.

We now show that A is nonsingular. Suppose Ad^T is the zero vector; where $d = (d_0, d_1, \cdots, d_n)$. Form $h(x) = d_n x^n + \cdots + d_1 x + d_0$. Clearly, $h^i(x)$ has a_i double zeroes and b_i single zeroes; $0 \leqq i \leqq k$. Applying the lemma, $h^k(x)$ has $s + (b_0 + \cdots + b_k) + 2(a_0 + \cdots + a_k) - k = n + 1 - k$ zeroes. As $h^k(x)$ has degree $n - k$ or less, $h^k = 0$.

Using (5), and $s + (b_0 + \cdots + b_k) + 2(a_0 + \cdots + a_k) = n + 1$, we have

$$s + (b_0 + \cdots + b_j) + 2(a_0 + \cdots + a_j) \geqq j + 2\ ; j = 0, 1, \cdots, k - 1\,.$$

Hence by (4), $h^j(x)$ has at least 2 zeroes, $j = 0, 1, \cdots, k - 1$. As $h^k = 0$, this shows that $h = 0$, or $d_i = 0$, $i = 0, 1, \cdots, k$. Thus, A is nonsingular.

As A is nonsingular, we can fit $f(x), f^1(x), \cdots, f^k(x)$ exactly on the $(b_0 + \cdots + b_k) + 2(a_0 + \cdots + a_k) \leqq n$ extreme points of $c - f$. That is, we can find $r(x)$ of degree n or less, so that if $|c^i(x') - f^i(x')| = ||c^i - f^i|| = M(c - f)$, then $r^i(x') - f^i(x') = 0$. It may well be, even though $r^i(x') - f^i(x') = 0$, that $|r^j(x') - f^j(x')| \geqq M(c - f)$ for some $j, j \neq i$. If this is the case, x' must not have been one of the $a_j + b_j$ extreme points of $c^j - f^j$. If $|c^j(x') - f^j(x')| < M(c - f)$, there is some $t \in (0, 1)$ such that

$$|t(r^j(x') - f^j(x')) + (1 - t)(c^j(x') - f^j(x'))| < M(c - f) .$$

As $E(c, f)$ was supposed to be a finite set, we can use the above remark to choose some $t \in (0, 1)$ such that

$$|t(r^i(x) - f^i(x)) + (1 - t)(c^i(x) - f^i(x))| < M(c - f) , \quad \text{for all } x \in E(c, f) ,$$

$i = 0, 1, \cdots, k$. This gives $tr + (1 - t)c$ a better approximation to f on $E(c,f)$ than is c. Thus, c could not have been a best approximation.

The proof above, except for cumbersome notational modifications, clearly establishes the more general

THEOREM 3. *Let $i, j, \cdots, k$ be any finite sequence of nonnegative integers, $i < j < \cdots < k$. Let $f(x)$ be $(k + 1)$-times differentiable on $[a, b]$. Among all polynomials $h(x)$ of degree n or less, let $p(x)$ be one that minimizes:*

$$\max \{||h^i - f^i||, ||h^j - f^j||, \cdots, ||h^k - f^k||\} .$$

If $q(x)$ is another such minimizing polynomial, then $q^k = p^k$.

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